Linear Relationships

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LINEAR RELATIONSHIPS

In a linear expression the powers of both variables is 1. A 'linear relationship' occurs when the variables are related to each other in a linear expression, for example $y = x - 3$. Each linear relationship has a unique straight line on the axes.

Answer these questions, *before* working through the chapter.

I used to think:

A line is made up by joining points together. These points are in the form (x, y). How can a linear relationship be used to find the points of a line?

What do parallel lines have in common?

Answer these questions, *after* working through the chapter.

But now I think:

A line is made up by joining points together. These points are in the form (x, y). How can a linear relationship be used to find the points of a line?

What do parallel lines have in common?

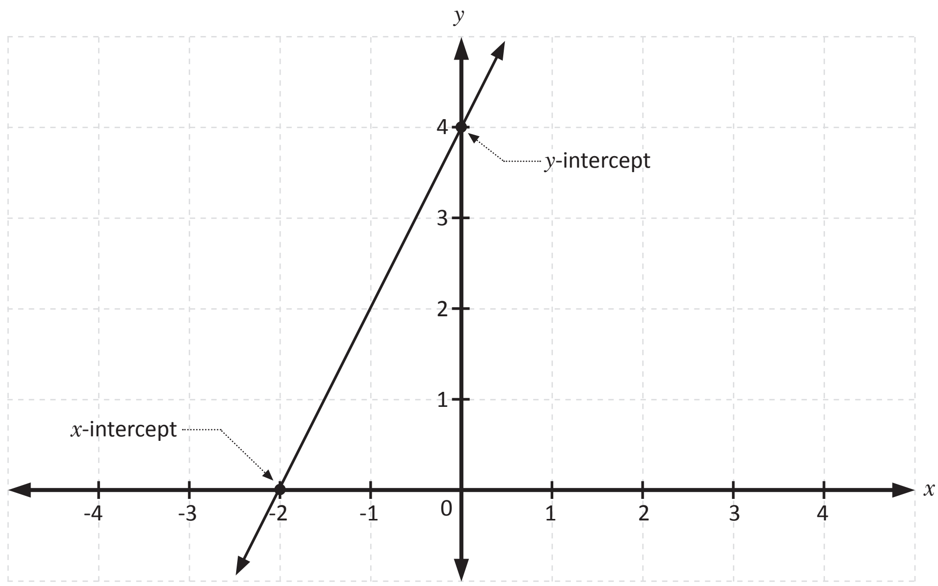
What do I know now that I didn't know before?

Intercepts

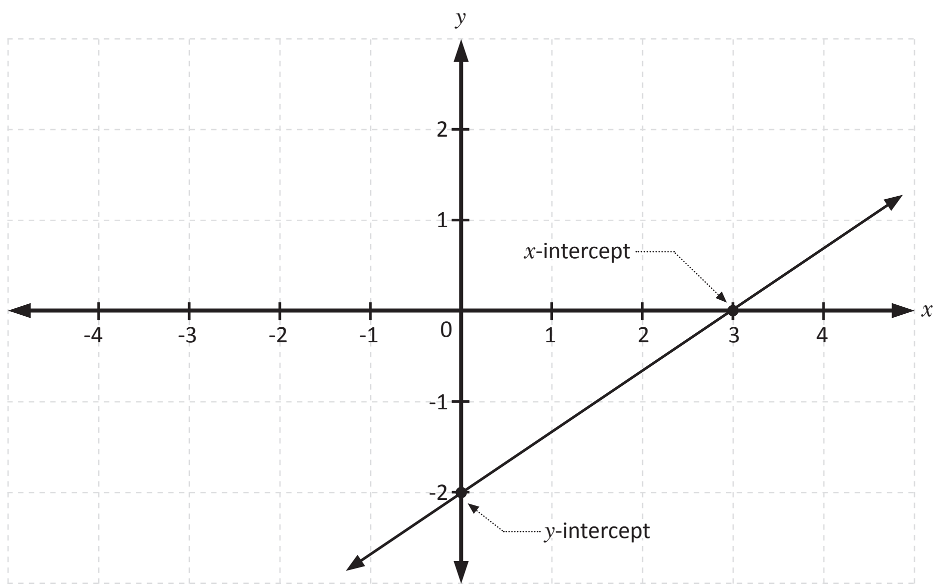
Straight lines have an x-intercept and a y-intercept. The x-intercept is the point where the line cuts the x-axis and the y-intercept is where the line cuts the y-axis. Here are two examples:

Find the intercepts of these lines:

a The x-intercept of the line is -2.
The y-intercept of the line is 4.



b The x-intercept of the line is 3.
The y-intercept of the line is -2.



Each straight line can be written by a linear equation with y and x. Mathematically it is said that each straight line is represented by a linear equation. This equation can be written in two ways (both are correct). They can be written in **gradient-intercept form** or in **general form**.

Gradient – Intercept form of a Line

Each line has a gradient, this is the slope of the line. The greater the gradient, the steeper the slope. The first line on the pervious page has a gradient of 2 and a y-intercept of 4. The equation of the line in gradient-intercept form is:

Gradient

$$y = 2x + 4$$

y-intercept

The equation is in gradient-intercept form because it depends on the gradient and the y-intercept.
In this form, y is always the subject of the equation. In general the equation of a straight line is:

$$y = mx + b$$

where m is the gradient of the line and b is the y-intercept.

Find the gradient and y-intercept of the line $3y - 9x = -15$

Make y the subject of the equation

$$3y = 9x - 15$$
$$y = 3x - 5$$

$m = 3 \quad b = 5$

The gradient is $m = 3$ and the y-intercept is $b = -5$.

Find the equation of the line with y-intercept $b = 4$ and passing through the points $(4, 5)$ and $(7, -4)$

Find the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 5}{7 - 4} = -3$$

Use the gradient and y-intercept: $y = mx + b = -3x + 4$

General Form of a Line

Each straight line can also be written as $ax + by + c = 0$ where a, b and c are integers and $a \geq 0$. In this form, all the terms are on one side and the coefficient of x is positive. This is called the General Form of the equation.

Write $4y = -7x + 8$ in (a) gradient-intercept form and (b) general form.

(a) Standard form: Make y the subject:

$$y = -\frac{7}{4}x + 2$$

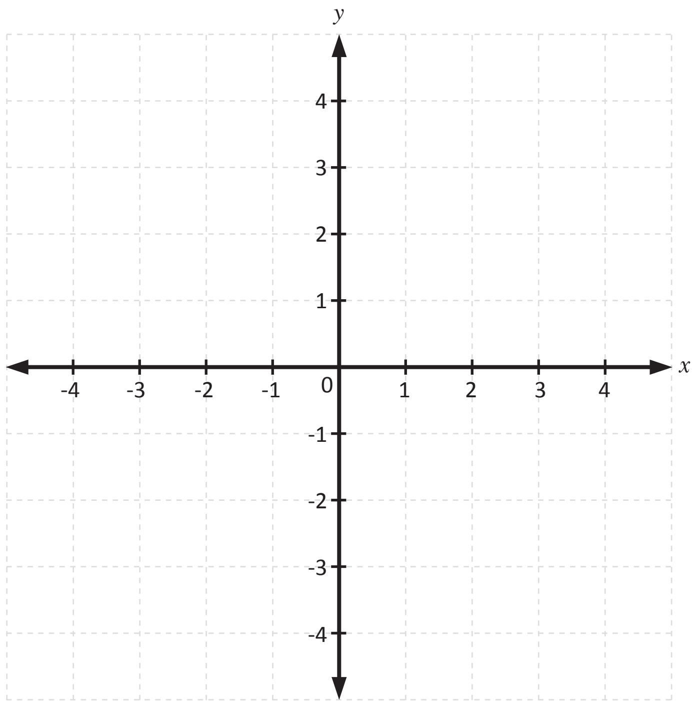
(b) General form: Move all the terms to one side:

$$7x + 4y - 8 = 0$$

1. Draw the following lines on the provided axes:

a) A line with x-intercept 2 and y-intercept -1.

b) A line with y-intercept 3 and x-intercept -4.



2. Write the following in gradient-intercept form:

a) $4x = 2y + 1$

b) $-y = x + 1$

c) $x + 2y - 6 = 0$

d) $3x = -9y$

3. Write in standard form the equation for a line with gradient $m = -3$ and y-intercept $b = 5$.

4. Write the following in general form:

a) $y = 3x - 7$

b) $5x = 2y - 1$

c) $y = 3 + \frac{x}{4}$

d) $-2x + 3y + 4 = 0$

5. Find the gradient of the line given by $12x + 4 = 8$:
(Hint: Write in standard form first)

6. Write the equation for a line with y-intercept $b = -2$ and gradient $m = 5$ in general form.

Determining if a Point is on a Line

If a point (x, y) is on a line then it will work in the line's equation.

Does $A(-2, -7)$ or $B(-2, 7)$ lie on the line $y = 2x - 3$?

Substitute $x = -2$ into $y = 2x - 3$

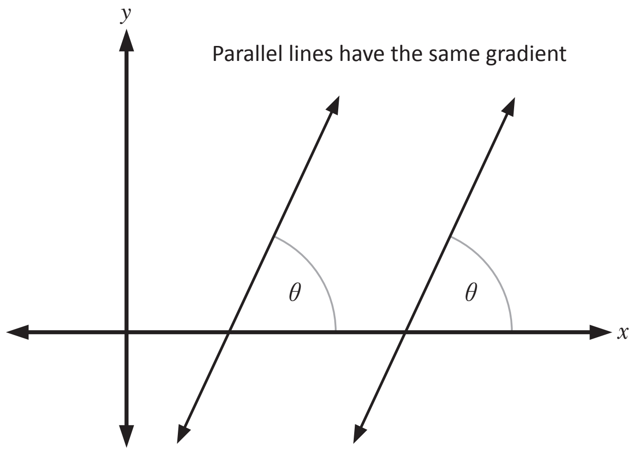
$y = 2(-2) - 3$

$y = -7$

$(-2, -7)$ lies on the line $y = 2x - 3$. This is point B.

Parallel Lines

Parallel lines have the same gradient. This is because parallel lines would make the same angle with the x-axis. Also, remember that $m = \tan\theta$. So if the angles are the same, then the lines have the same gradient. Here is an example.



Which of the following lines are parallel?

- Line 1: $y - 3x = 1$
- Line 2: $2y + 2 = 6x$
- Line 3: $y - 6x + 4 = 0$

Rewrite each line in standard form:

- Line 1: $y = 3x + 1$
- Line 2: $y = 3x - 1$
- Line 3: $y = 6x - 4$

Line 1 and Line 2 are parallel since they have the same gradient $m = 3$.

Four points have the coordinates $A(7, -4)$, $B(2, 6)$ and $C(4, -3)$. If $D(1, y)$, solve for y so that AB is parallel to CD.

Let m_1 be the gradient of AB and let m_2 be the gradient of CD.
Parallel lines have equal gradients so $m_1 = m_2$.

$$m_1 \rightarrow \frac{6-(-4)}{2-7} = \frac{y-(-3)}{1-4} \leftarrow m_2$$

$$-2 = \frac{y+3}{-3}$$

$$y = 3$$

So D has coordinates $D(1, 3)$.

1. Which of the 2 points $(-1,6)$ or $(-1,5)$ lie on the line $y=-x+4$?

2. Find any possible values for x and y if the point (x,y) lies on the line $y=3x+7$.

3. Find any possible values for x and y if the point (x,y) lies on the line $4y-16x+12=0$.

4. Solve for x if the point $(x,9)$ lies on the line $2y-10x+2=0$.

5. Are these lines parallel?

a) $2x + 2y = 2$ and $2y = -2x + 3$

b) $y = 3x + 2$ and $y + 3x = -5$

c) $y = 2x - 3$ and $6x + 3y - 9 = 0$

d) $y - 2x + 6 = 0$ and $4y = 8x + 1$

6. Find the value of x if the line passing through $(5, 10)$ and $(x, 4)$ is parallel to $y = 6x + 7$.

7. If a line has y-intercept 4 and is parallel to $y = -5x - 3$, then what is the equation of the line?

Graphing Straight Lines

Graphing straight lines is simply drawing them in the correct place on the axes. Each straight line is based on an equation.

Graph the equation $y = 2x - 1$

Step 1: Use a table to find y-values for any 3 test x-values. This example uses -1, 0 and 1, but any three values will work.

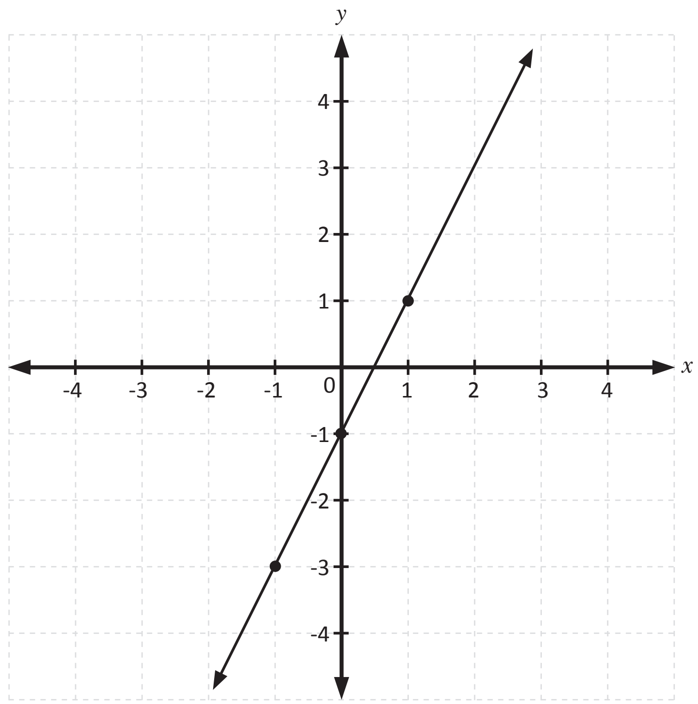
Test values

x	-1	0	1
y	-3	-1	1

$y = 2x - 1$

Using the table, the line passes through the 3 points $(-1, -3)$, $(0, -1)$ and $(1, 1)$.

Step 2: Plot these 3 points on the axis and draw the line which goes through them.



Why 3 test values?

You may be wondering why we find 3 points since only 2 points are necessary to draw a straight line. The third point is used to check for mistakes. If three points are found, and a single straight line can't be drawn through all 3 points then a mistake was made somewhere.

What if we use different test values?

What happens if we choose different test values for x with the same equation? Will the line change?

Redo the previous example with test values $-\frac{1}{2}, \frac{1}{2}$ and 2 line.

x	$-\frac{1}{2}$	$\frac{1}{2}$	2
y			

What is the result?

As you can see, the new line is the same as the original line. It doesn't matter which values we choose as test x-values. So, you should always choose values that are easy to work with.

Draw the graph of the equation $6x + 3y - 6 = 0$

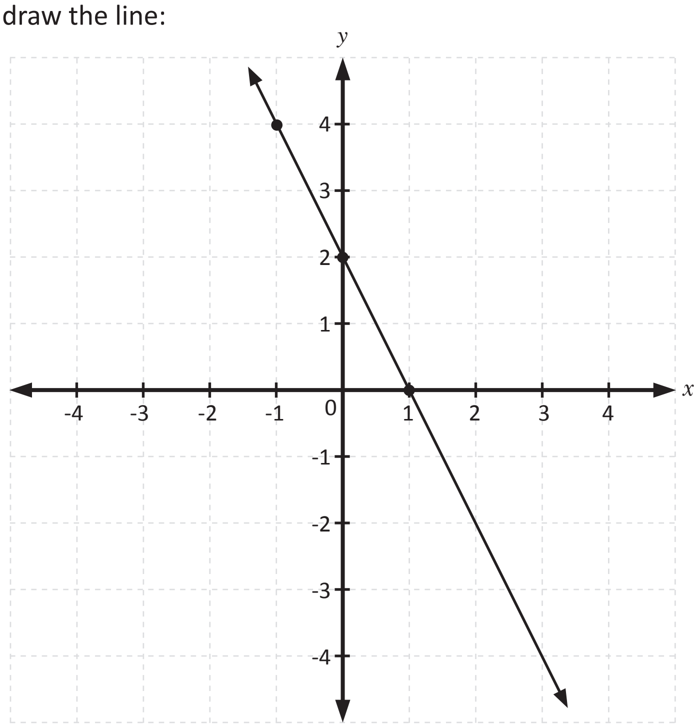
Step 1: Convert equation to standard form:

$$y = -2x + 2$$

Step 2: Use a table of values:

x	-1	0	1
y	4	2	0

Step 3: Plot the points and draw the line:



We can do another check based on the gradient.
If m is positive then the line leans to the right. If m is negative the line should lean to the left.

Horizontal and Vertical Lines

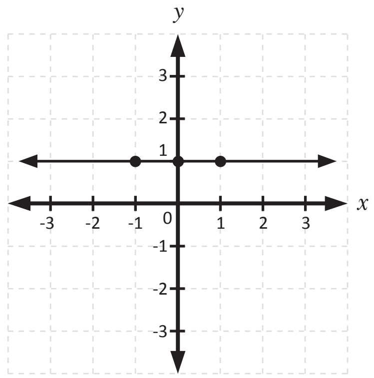
For horizontal lines $m = 0$. So $y = 0x + b = b$. Thus the value of y remains constant no matter the value of x.

Draw the graph $y = 1$

x	-1	0	1
y	1	1	1

← y is constant for different values of x

Using the table, the line passes through $(-1, 1)$, $(0, 1)$ and $(1, 1)$.



The rule for graphing horizontal lines is that the line $y = b$ will cut the y-axis at b. You could even say that the equation of the x-axis is $y = 0$, since it is a horizontal line cutting the y-axis at 0.

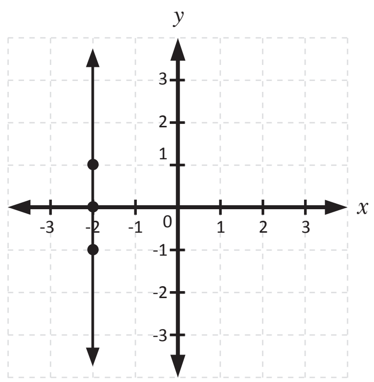
Vertical lines are different.
The formula $y = mx + b$ can't be applied to vertical lines because the gradient is undefined. Equations for vertical lines take the form $x = a$. The x-value remains constant no matter what the value of y is.

Draw the graph $x = -2$

x	-2	-2	-2
y	-1	0	1

← x is constant for different values of y

Using the table, the line passes through $(-2, -1)$, $(-2, 0)$ and $(-2, 1)$

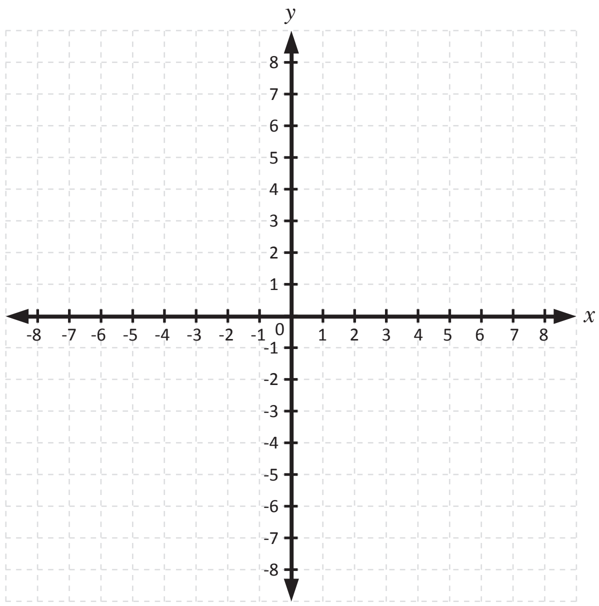


The rule for graphing vertical lines is that the line $x = a$ will cut the x-axis at a.
You could even say that the equation of the y-axis is $x = 0$.

1. Draw the following graphs using the table method:

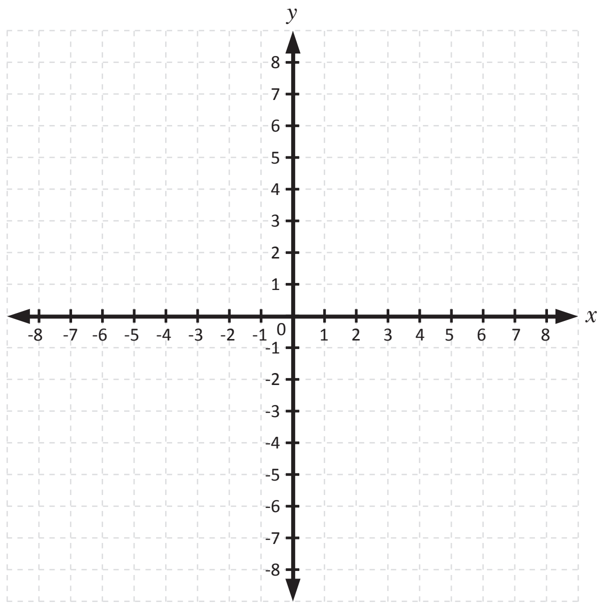
a) $y = 6x$

x			
y			



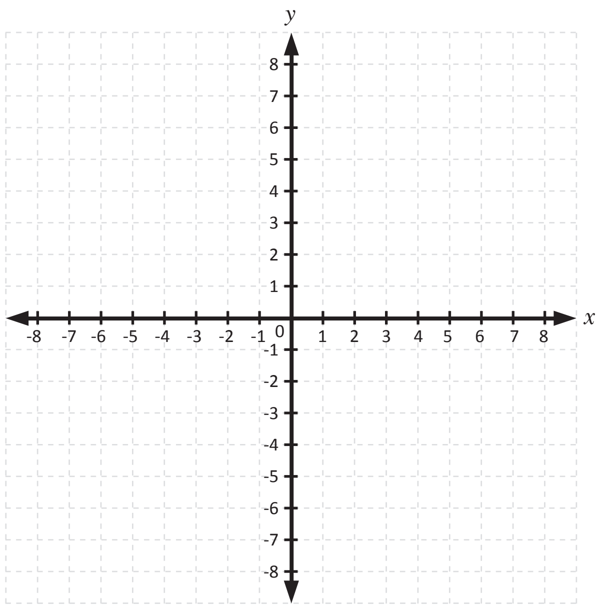
b) $y = 7x - 2$

x			
y			



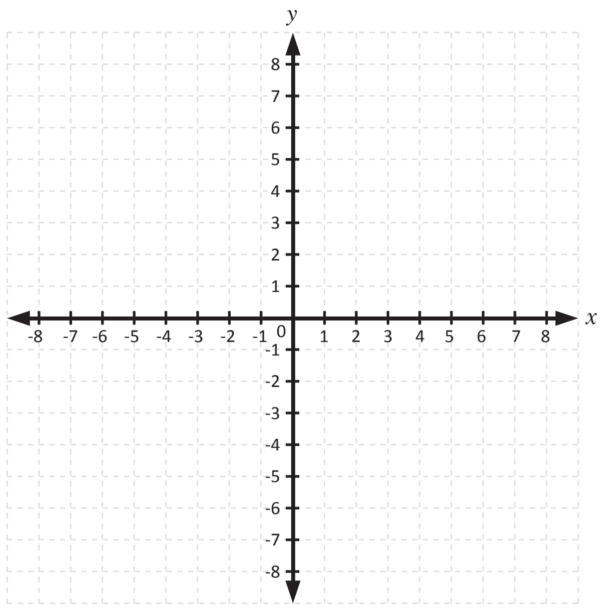
c) $3x - y + 4 = 0$

x			
y			



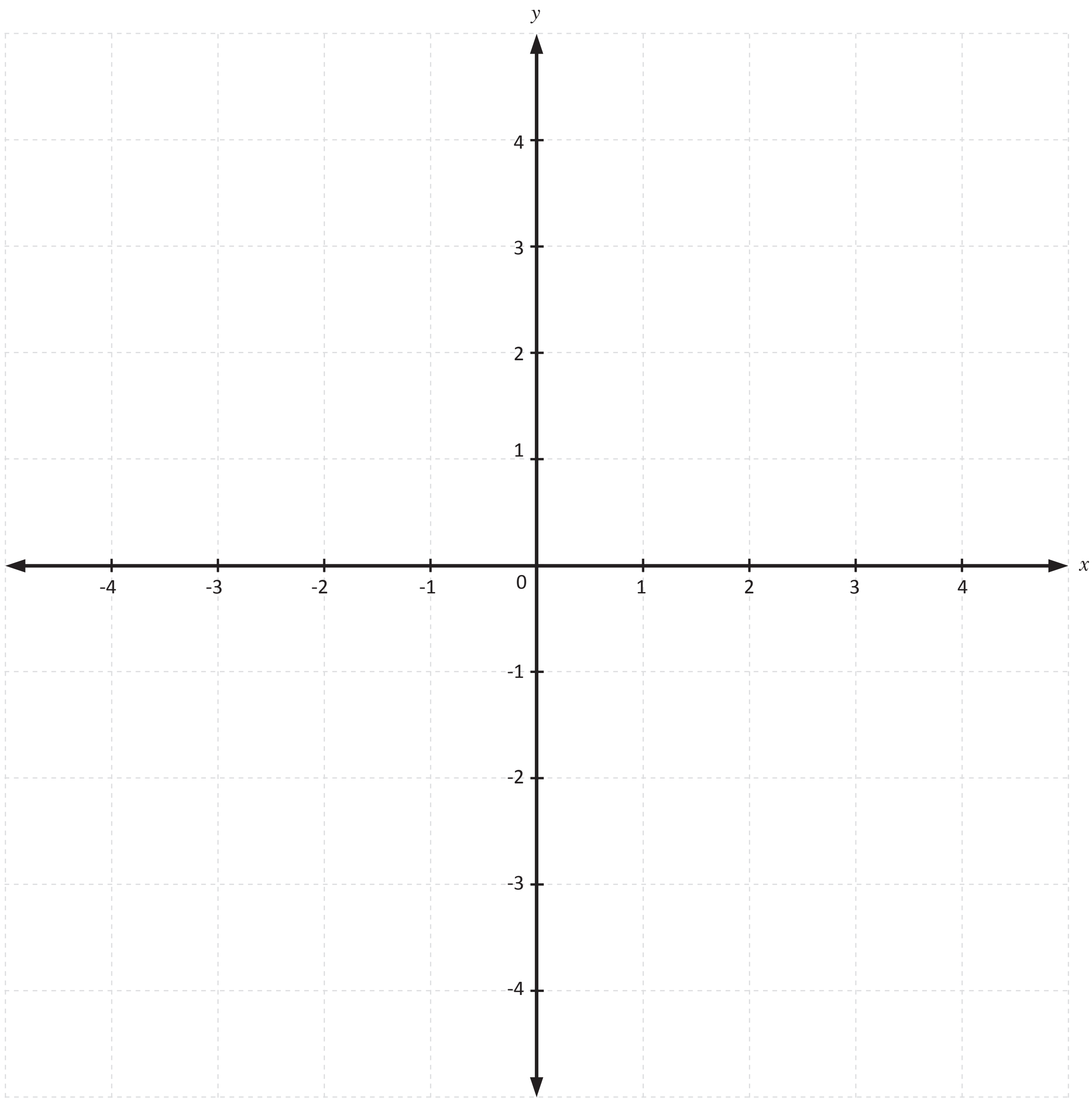
d) $y = \frac{1}{2}x - 4$

x			
y			

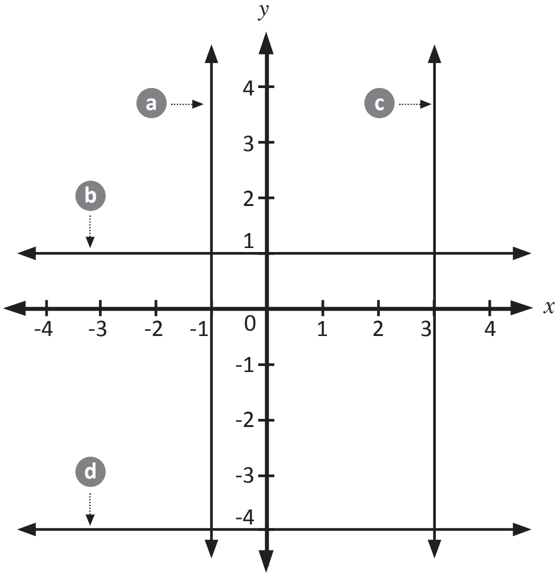


2. Draw the following lines on a number plane:

a $x = 2$ b $y = -3$ c $y = 4$ d $x = -4$



3. Write the equations of the following lines:



a ____________ b ____________ c ____________ d ____________

4. Write down the coordinates where lines a and d – from the above question – intersect each other.

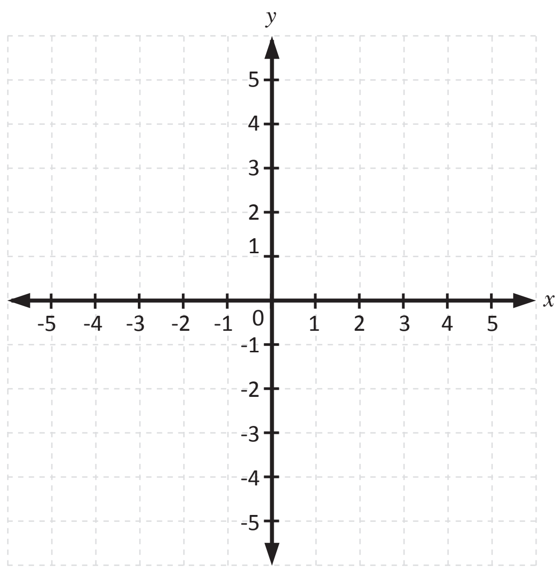
5. Find the equations of the following lines:

A vertical line passing through points $(-1, 5)$ and $(-1, -2)$.

A horizontal line passing through $(0, 3)$.

A line parallel to the x-axis and passing through $(4, 2)$.

A line parallel to the y-axis and passing through $(-3, 1)$.



6. Draw the following graphs on the same set of axes:

a $y = -x$

x			
y			

b $y = -2x$

x			
y			

c $y = x + 2$

x			
y			

d $y = 2x$

x			
y			

7. What do you notice about the lines as the value of m increases in their equations?

8. Draw these lines on the same set of axes below:

a $y = 6x$

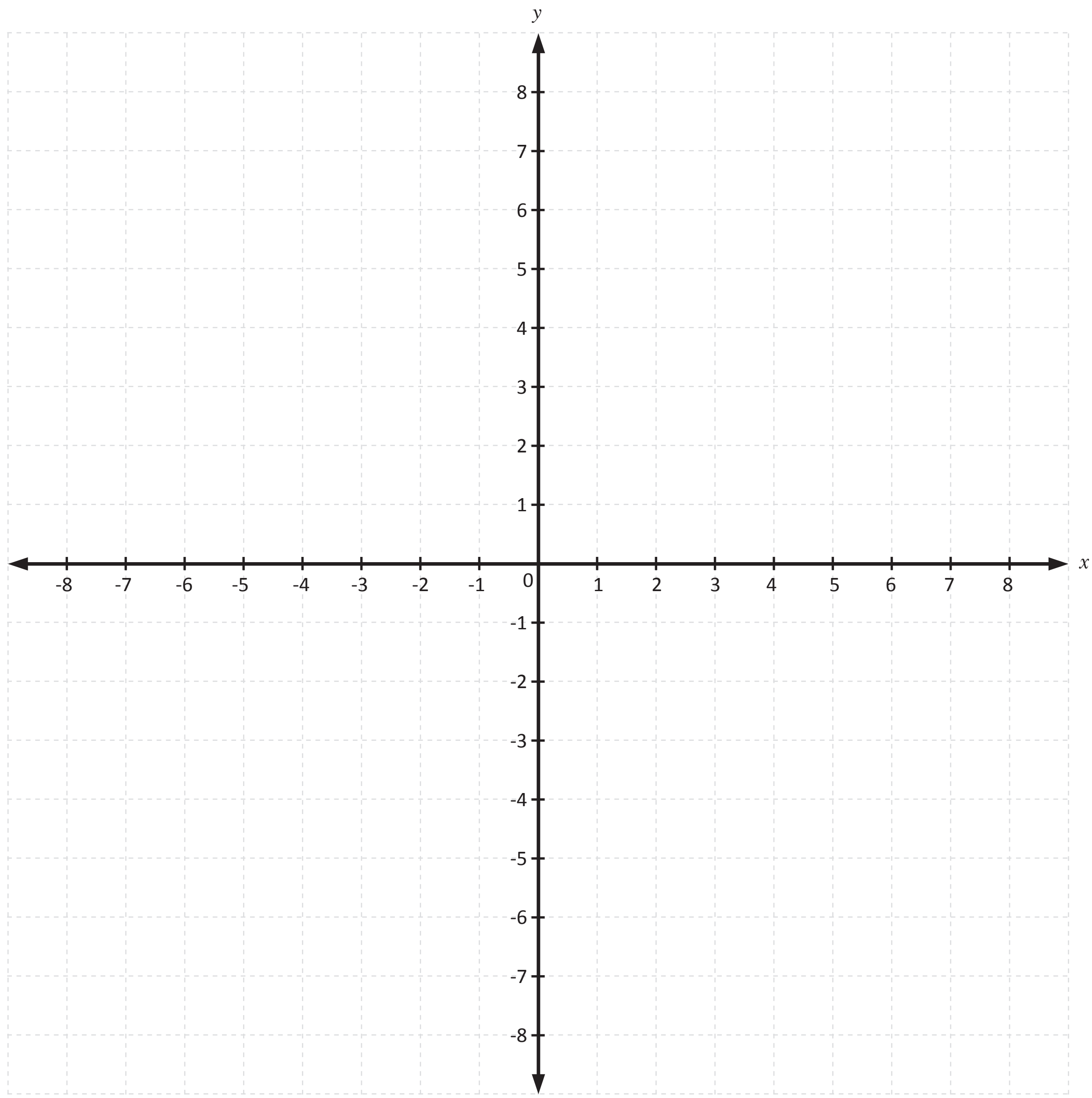
x			
y			

b $y = 7x - 2$

x			
y			

c $4x - y = -3$

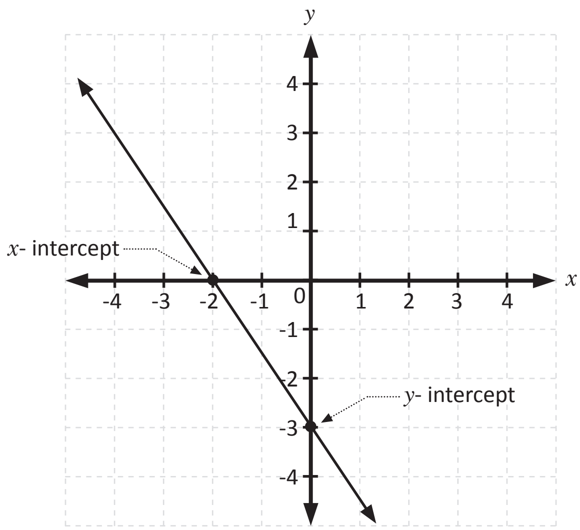
x			
y			



9. Is this what you were expecting from these lines? Why?

Each line has (at most) one x and y-intercept. If we know what those intercepts are, then we can graph a line very easily.

Draw a line which has x-intercept -2 and y-intercept -3



The next question is: How do we find those intercepts from the equation of the line?

Using Intercepts to Graph Straight Lines

If we can find the intercepts, then we can draw the line without a table.

- To find the x-intercept, set $y = 0$ and solve for x
- To find the y-intercept, set $x = 0$ and solve for y

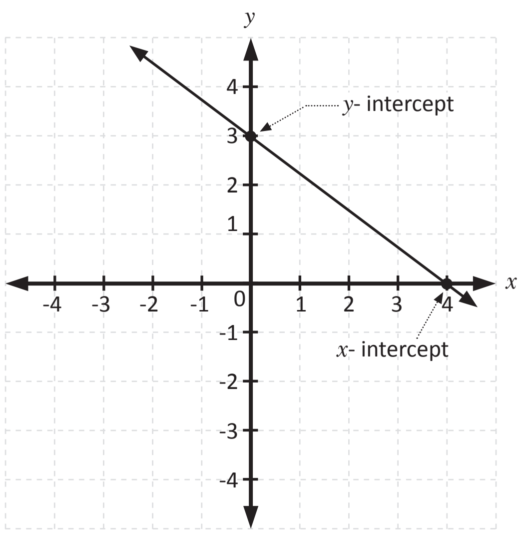
For equation $y = 3x + 4y - 12 = 0$, find the x and y-intercept and use them to draw the straight line

- For x-intercept, set $y = 0$: $3x + 4(0) - 12 = 0$ ← $y = 0$

$$3x = 12$$
$$x = 4$$

- For y-intercept, set $x = 0$: $3(0) + 4y - 12 = 0$ ← $x = 0$

$$4y = 12$$
$$y = 3$$



Intersection of Lines

When lines cross through each other, the point of crossing is called the point of intersection. This point lies on both lines. There are two methods to find this point of intersection.

Method 1: Reading the point of the graph

In this method we need to draw both graphs. Simply graph the two lines and read the point of intersection from the graph.

Find the point of intersection of $y = x + 6$ and $y + 2\ x = 3$

To draw $y = x + 6$, find

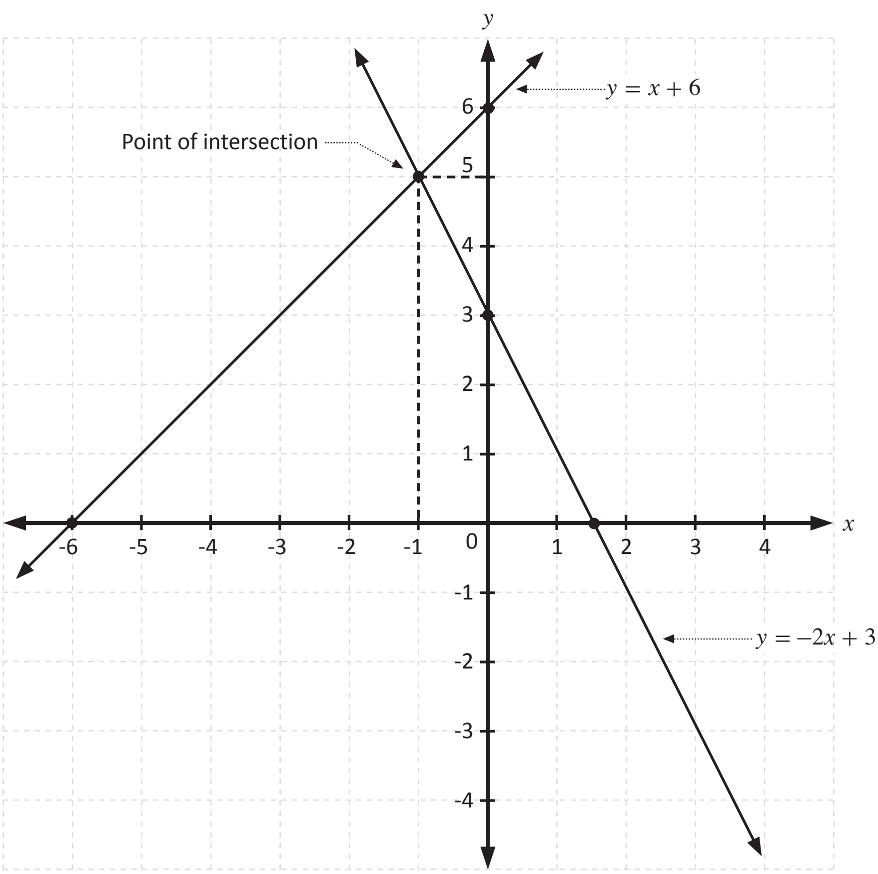
the intercepts of $y = x + 6$:

- x-intercept: $0 = x + 6$
  $x = -6$
- y-intercept: $y = 6$

To draw $y = -2x+3$, find

the intercepts of $y = -2x+3$:

- x-intercept: $0 = 2x - 3$
  $x = \frac{3}{2}$
- y-intercept: $y = 3$



From the above diagram, the point of intersection can be read off the graph as $(-1,\ 5)$.

Method 2: Set the equations equal to each other and solve.

In this method we do not need to draw any graphs.

Find the point of intersection of $y = x + 6$ and $y + 2x = 3$

Step 1: Ensure both equations are in gradient-intercept form.

$y = x + 6$ and $y = -2x + 3$

Step 2: Set the right sides of the equations equal to each other and solve for x.

$$\begin{aligned} x + 6 &= -2x + 3 \\ 3x &= -3 \\ x &= -1 \end{aligned}$$

Step 3: Substitute this x-value into either of the equations (it doesn't matter which one) to find the y-value.

$y = (-1) + 6 = 5$

Step 4: Identify the point where the lines intersect.

Since $x = -1$ and $y = 5$, the point of intersection is $(-1, 5)$

It is even easier when one of the lines is vertical or horizontal.

Find the point of intersection of $x = -6$ and $y = \frac{1}{3}x + 4$

Since the first line is vertical, we know that the x-coordinate of the intersection is $x = -6$, (as it is given).

(We don't have to solve for x).

Substitute the x-value into the equation of the other line to find the y-value.

$$\begin{aligned} y &= \frac{1}{3}(-6) + 4 \\ y &= 2 \end{aligned}$$

The point of intersection is $(-6, 2)$

Find the point of intersection of $y = 9$ and $y = 4x - 3$

Since the first line is horizontal, we know that the y-coordinate of the intersection is $y = 9$.

Substitute the y-value into the equation of the other line to find the x-value.

$$\begin{aligned} 9 &= 4x - 3 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

The point of intersection is $(3, 9)$.

1. Find the intercepts of these lines:

a $y = 2x - 4$

b $x + y = -7$

c $2x - y + 18 = 0$

d $x - 3y - 21 = 0$

2. Draw the graph of a line with these intercepts:

a x-intercept -2 and y-intercept 4

b x-intercept 6 and y-intercept -3

3. Use the intercept method to sketch the graphs of the following equations:

a) $y = 3x - 9$

b) $6y + 12x + 30 = 0$

4. Graph each pair of lines on the same axis to find the point of intersection:

a $x = 1$ and $y = 8x - 8$

b $y = x - 5$ and $2x + y + 4 = 0$

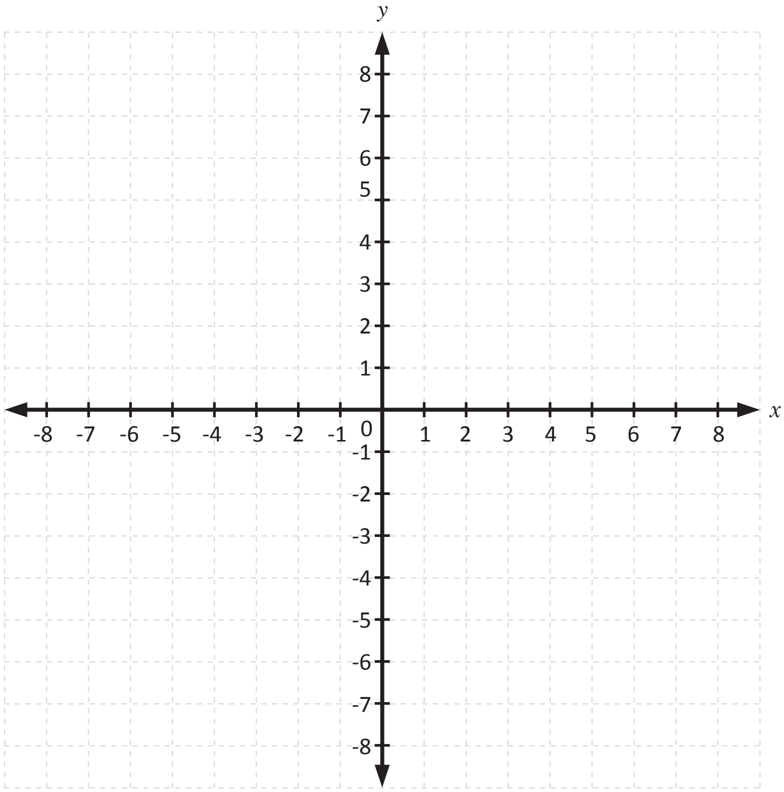
5. Find the point of intersection without drawing any graphs.

a $y = -2$ and $x - 2y - 11 = 0$

b $y = 5x - 8$ and $6x + 2y - 20 = 0$

6. Draw the following lines on the same set of axes:

- Line 1: $y = 3x + 6$
- Line 2: $y = 3x - 6$
- Line 3: $y = -2x + 8$



a What is the point of intersection of Line 1 and Line 3?

b Will Line 1 and Line 2 intersect at any point?

c Why do you think this is so?

7. Identify whether the following pairs of lines will intersect or not.

a) $y = 4x + 2$ and $y = 4x - 7$

b) $y = 2x + 2$ and $x = -2x - 7$

c) $x + y = 7$ and $y = x + 2$

d) $3x + 4y + 3 = 0$ and $6x + 8y + 5 = 0$

8. Use substitution to prove that $y = \frac{1}{2}x + 13$ and $y = 4x - 1$ intersect at the point $(4,15)$.

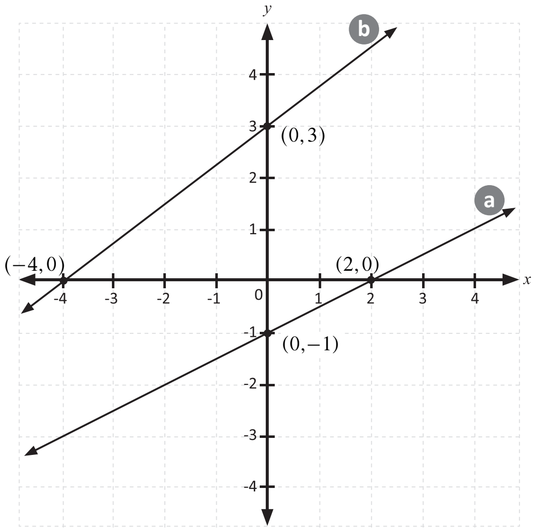
(Hint: Show the point of intersection is on both lines)

9. What is the point of intersection of the lines $y = -3$ and $x = 17$?

10. Find the equation of the horizontal line, exactly in the middle of $y = -4$ and $y = 6$.

Basics:

1. (a) and (b)



2. (a) $y = 2x - \frac{1}{2}$ (b) $y = -x - 1$

(c) $y = \frac{-x}{2} + 3$ (d) $y = \frac{-1}{3}x + 0$

3. The standard form of the equation is $y = 3x + 5$.

4. (a) $3x - y - 7 = 0$ (b) $5x - 2y + 1 = 0$

(c) $x - 4y + 12 = 0$ (d) $2x - 3y - 4 = 0$

5. The gradient (m) is -3

6. $5x - y - 2 = 0$

Knowing More:

1. The point $(-1, 5)$ is on the line $y = -x + 4$.

2. The point $(100, 307)$ works for the line $y = 3x + 7$, which shows that the arithmetic is correct.

3. The point $(2, 5)$ works for the line $4y - 16x + 12 = 0$, which shows that the arithmetic is correct.

4. $x = 2$

5. (a) Not parallel (b) Not parallel

(c) Not parallel (d) Parallel

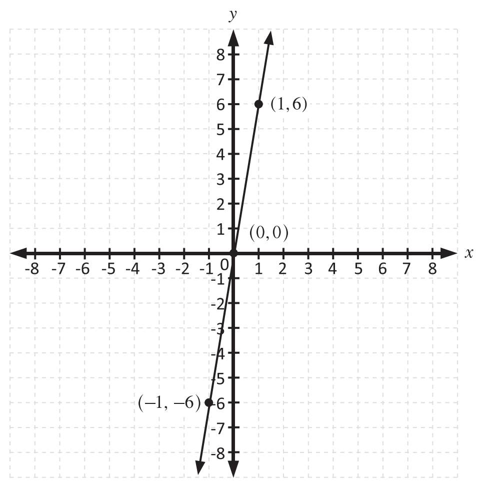
6. $x = 4$

7. $y = -5x + 4$

Using Our Knowledge:

1. (a)

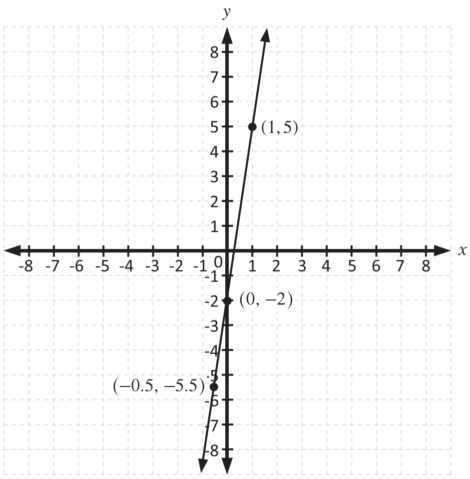
x	-1	0	1
y	-6	0	6



Using Our Knowledge:

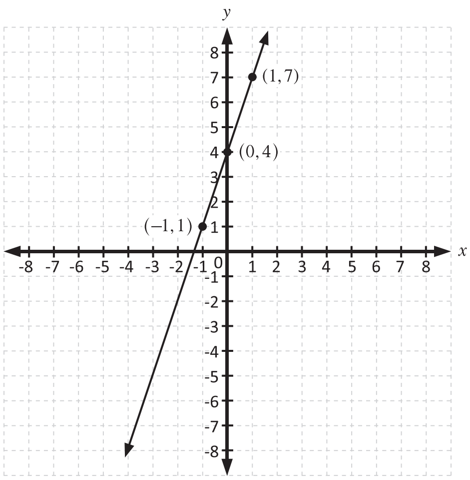
1. **b**

x	-0.5	0	1
y	-5.5	-2	5



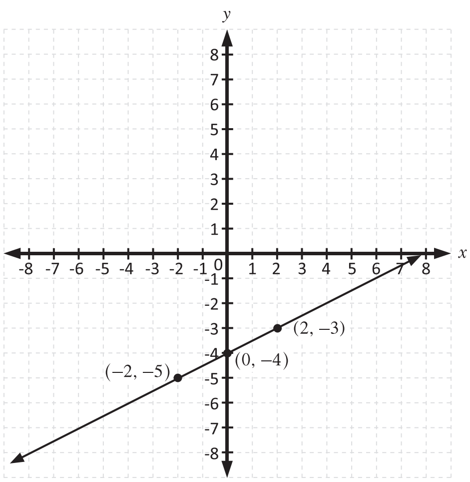
c

x	-1	0	1
y	1	4	7



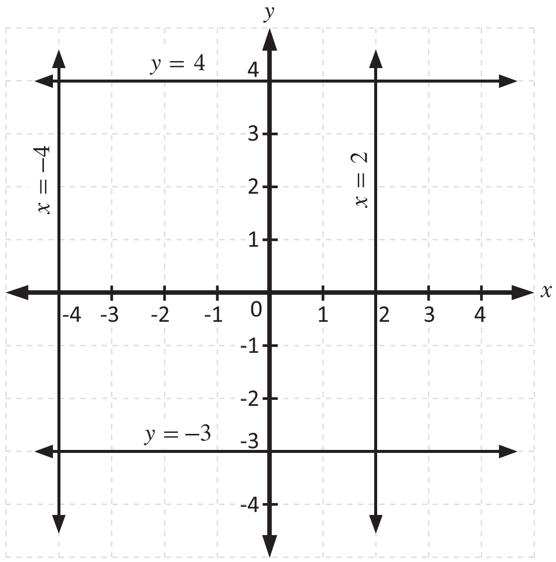
d

x	-2	0	2
y	-5	-4	-3



Using Our Knowledge:

2.

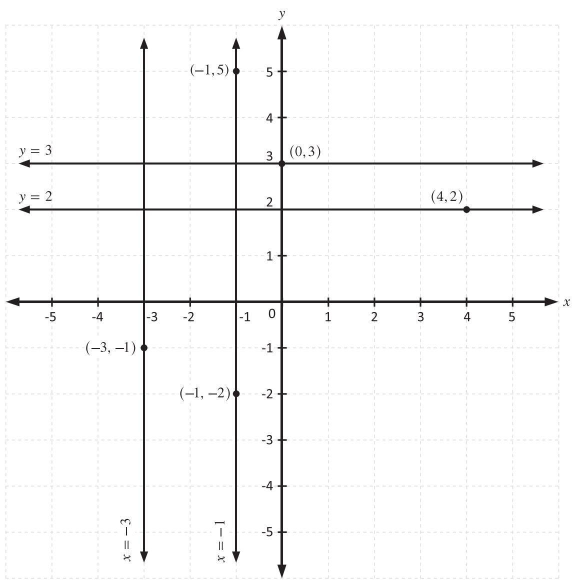


3. **a** $x = -1$ **b** $y = 1$

c $x = 3$ **d** $y = -4$

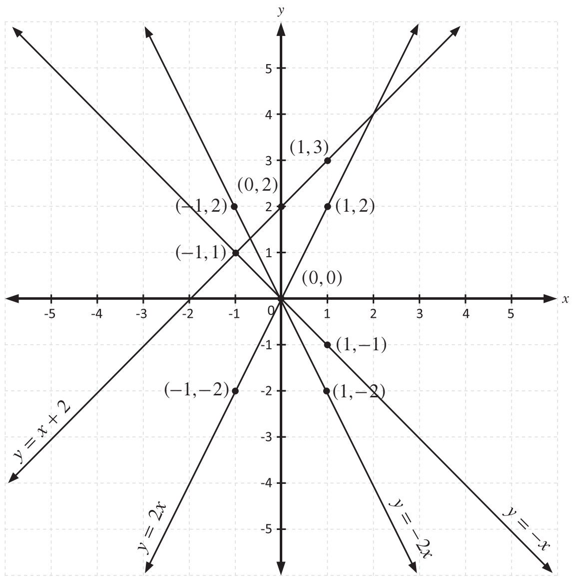
4. Line **a** and **d** intersect at $(-1, -4)$

5.



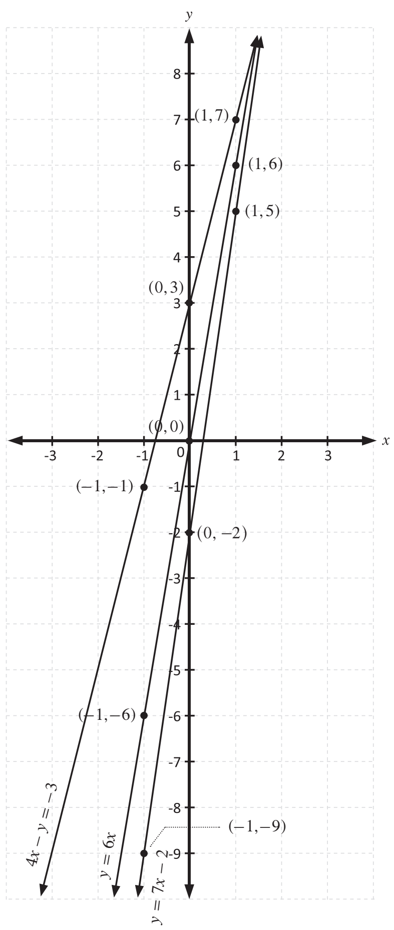
Using Our Knowledge:

6.



7.
- If m is positive, the line moves from bottom left to top right
- If m is negative, the line moves from bottom right to top left

8.



Thinking More:

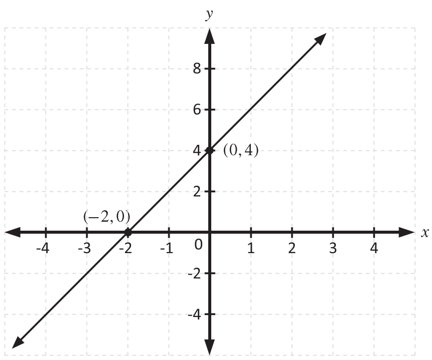
1.
a. The x-intercept is $(2, 0)$ and y-intercept is $(0, -4)$.

b. The x-intercept is $(-7, 0)$ and y-intercept is $(0, -7)$.

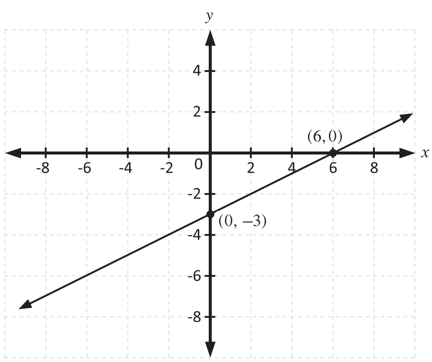
c. The x-intercept is $(-9, 0)$ and y-intercept is $(0, 18)$.

d. The x-intercept is $(21, 0)$ and y-intercept is $(0, -7)$.

2. a

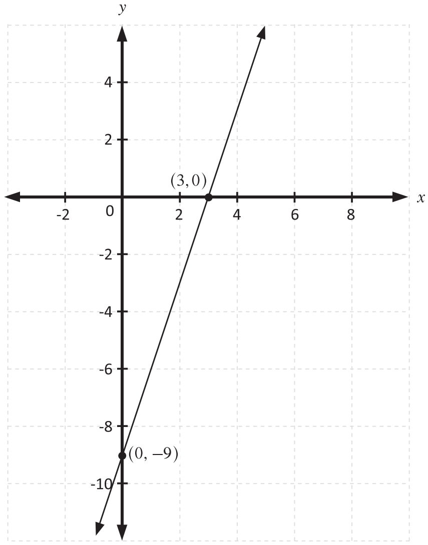


b

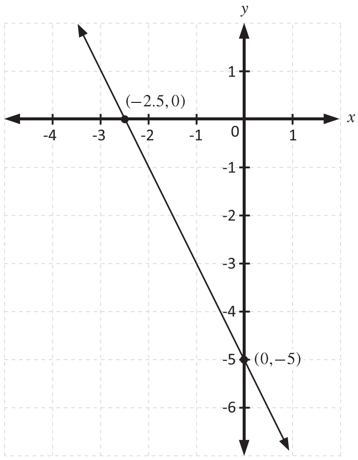


Thinking More:

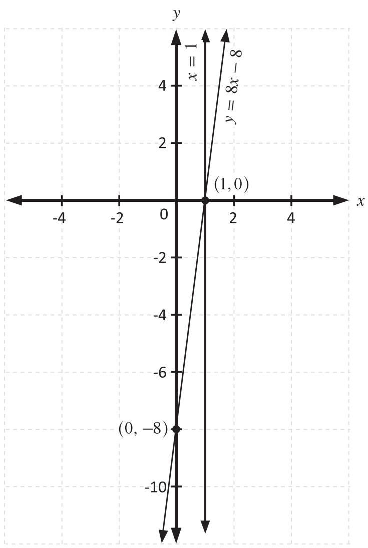
3. a



b



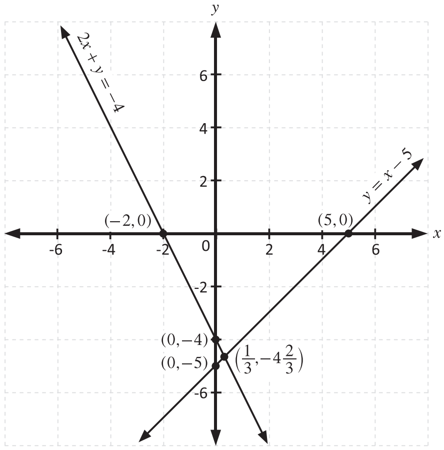
4. a



The point of intersection is $(1, 0)$.

Thinking More:

4. b



The point of intersection is $(\frac{1}{3}, -4\frac{2}{3})$.

5. a The point of intersection is $(7, -2)$.

b The point of intersection is $\left(\frac{9}{4}, \frac{13}{4}\right)$.

6. a The point of intersection of Line 1 and Line 3 is $(0.4, 7.2)$.

b No.

c Line 1 and Line 2 have the same gradient. This means they are parallel and will not intersect at any point.

7. a These have the same gradient (4) so they will not intersect.

b These have different gradients $(2$ and $-2)$ so they will intersect.

c The gradients are different $(-1$ and $1)$ so the lines will intersect.

d These lines have the same gradient $-\frac{3}{4}$ so they will not intersect.

Thinking More:

9. The first line is horizontal with all y-value being -3. The second line is vertical with all x-values being 17, so the point of interaction is $(17, -3)$.

10. The line exactly in the middle of $y = -4$ and $y = 6$ is $y = 1$.